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ORDER, DISORDER, AND PHASE TRANSITION IN CONDENSED SYSTEM

T_c of Disordered Superconductors Near the Anderson Transition[1]

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Abstract—According to the Anderson theorem, the critical temperature T_c of a disordered superconductor is determined by the average density of states and does not change at the localization threshold. This statement is valid under assumption of a self-averaging order parameter, which can be violated in the strong localization region. Stimulating by statements on the essential increase of T_c near the Anderson transition, we carried out the systematic investigation of possible violations of self-averaging. Strong deviations from the Anderson theorem are possible due to resonances at the quasi-discrete levels, resulting in localization of the order parameter at the atomic scale. This effect is determined by the properties of individual impurities and has no direct relation to the Anderson transition. In particular, we do not see any reasons to say on "fractal superconductivity" near the localization threshold.

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1. INTRODUCTION

The general picture of coexistence of superconductivity and the Anderson localization was formed in the papers by Bulaevskii and Sadovskii [1–5] (see also [6, 7]). According to the Anderson theorem [8], the critical temperature T_c of a disordered superconductor is determined by the average density of states and does not depend on the form of one-particle eigenstates. Since the average density of states does not have singularity at the Anderson transition, so T_c has the analogous behavior. The coefficient of the gradient term in the Ginzburg–Landau expansion, determining the superconducting response of the system, remains finite at the critical point. In the localized phase, the system breaks up into quasi-independent blocks of size ξ (ξ is the localization length) and superconductivity is suppressed due to the size effect, when the average level spacing in such a block becomes greater than T_c.

Recently it was stated by Feigelman et al. [9, 10] that T_c increases at approaching the Anderson transition from the metallic side and continues to grow in the localized phase (with a maximum in the deep of it); it is related with multifractality of wave functions. More than that, T_c depends on the Cooper interaction constant g not exponentially, but in the power-law manner. Formally, this statement does not contradict to the papers [1–5]. Indeed, the Anderson theorem is valid under assumption of a self-averaging character of the order parameter, which in fact reduces to its spatial uniformity. According to estimates of [3, 4], the self-averaging property tends to violate when the localization threshold is approached and the space-inhomogeneous superconductivity is expected in the deep of the localization phase; so the true T_c can be greater than its value given by the Anderson theorem. In fact, controversy between the papers [1–5] and [9, 10] has an ideological character. The authors of [1–5] proceed from the standpoint that localization counteracts to superconductivity, so the latter encounters a lot of problems in the localized phase [2, 5]. Contrary, the growth of T_c after the mobility edge [9, 10] indicates that superconductivity not only "survives" but even "prospers" in the localized phase. It looks suspicious from the physical viewpoint and contradicts the experimental situation, which is in complete agreement with [1–5].

The present paper has an aim to clarify a situation. In fact, the essence of the problem is: how and in what extent self-averaging of the order parameter can be violated? The efficient approach to such problems was developed in [11–14] and consists in the study of individual defects and their influence on the transition temperature. In particular, for the plane defects arranged perpendicularly to the z axis with the period L along it, the change of T_c is determined by the formula[2]

$$\frac{\delta T_c}{T_{c0}} = \frac{1}{L\lambda_0^3} g^2 \int dz [\nu_0 \nu_1(z) + \nu_1(z)^2], \qquad (1)$$

if there are no surface states localized near defects. Here, g is the Cooper interaction constant, $\nu_1(z)$ is a deviation of the local density of states $\nu(z)$ from its unperturbed value ν_0, $\lambda_0 = g\nu_0$ is the dimensionless

[1] The article was translated by the author.

[2] We accept for simplicity that a plane defect changes only the density of states. Generalizations of (1), accounted for the change of the interaction constant g [12, 13] and the cut-off frequency ω_0 [14] are also available.

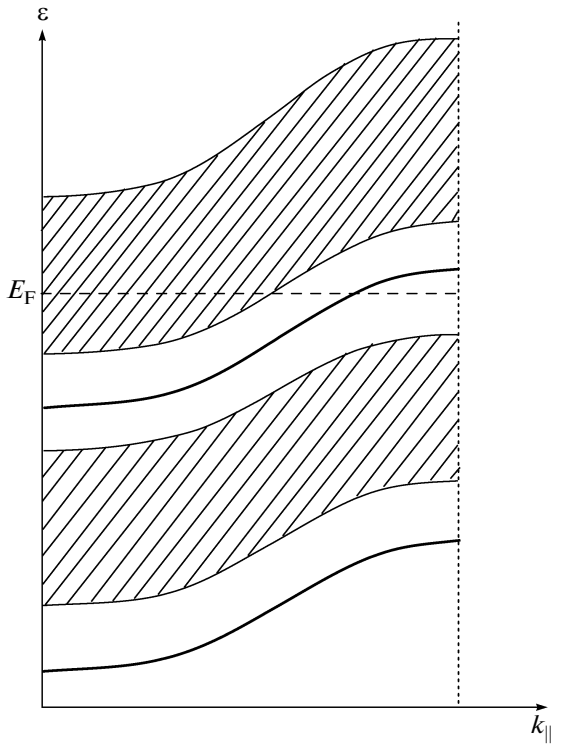

Fig. 1. If plane defects are arranged perpendicular to the z axis, the problem allows a separation of variables and the longitudinal quasi-momentum $k_{||}$ can be introduced. For fixed $k_{||}$, the spectrum is a set of bands with discrete levels splitted from them; if dependence on $k_{||}$ is taken into account, these levels turn into 2D bands, which can appear at the Fermi level.

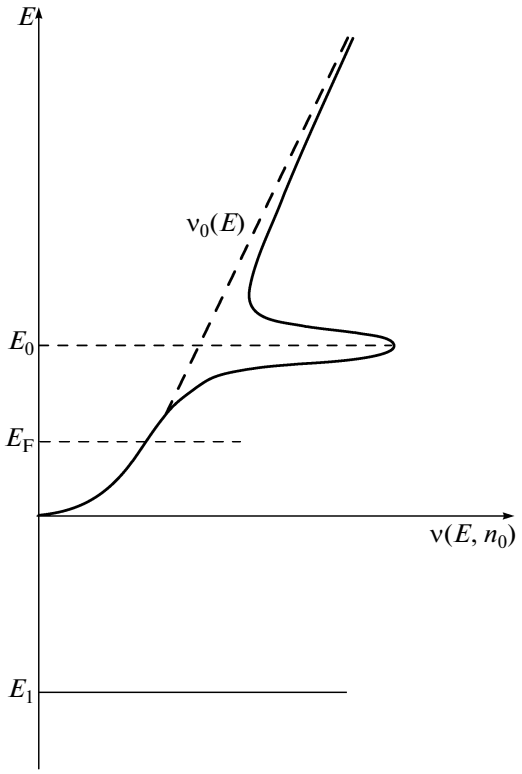

Fig. 2. A strong point defect inserted into site n_0 of an ideal lattice leads to appearance of the local (E_1) and quasi-local (E_0) levels. The latter corresponds to the maximum of the local density of states $\nu(E, n_0)$.

coupling constant, T_{c0} is the transition temperature in the absence of defects, integration is carried out over a vicinity of the single defect. For weak defects, only the linear in $\nu_1(z)$ term is essential, which exactly corresponds to the Anderson theorem and relates the change in T_c with the change of the average density of states. Generally, $\nu_1(z)$ is comparable with ν_0 and already Eq. (1) predicts a possibility of essential violation of the Anderson theorem. It is related with the fact that the initially uniform order parameter is influenced by strong defects and can increase or decrease in their vicinity. More essential violations of the Anderson theorem are possible, if the surface states localized near defect appear at the Fermi level (Fig. 1). In this case, the order parameter can be strongly localized near the plane defects, so T_c does not depend on L and is determined by the BCS formula $T_c = 1.14\omega_0\exp(-1/\lambda_{2D})$ with the coupling constant λ_{2D}, corresponding to the separated two-dimensional band (Fig. 1). A crossover between two regimes is appeared to be very sharp and the intermediate situation is of little interest. Formally, the described results correspond to the periodical arrangement of defects, but their character shows that the assumption on periodicity is not essential; so they give a complete picture for the small "impurity" concentration in the 1D geometry.

Analogous effects are possible in case of the point defects, where the localized regime for the order parameter is related with existence of the quasi-local states (Fig. 2). A detailed investigation of these effects allows to obtain the complete picture of possible violations of self-averaging. The main conclusion is that such violations are determined by individual defects and have no direct relation to the Anderson transition. One can distinguish two typical situations.

If disorder is created by weak impurities (Fig. 3a), then the assumption of self-averaging is always true and the Bulaevskii-Sadovskii picture is literally applicable. The mobility edge lies near the initial band edge and T_c is falling quickly at approaching it from the metal side due to decrease of the density of states; hence, superconductivity becomes practically unobservable before the mobility edge is reached. Such situation is typical for the traditional superconductors, which are good metals and effectively screen any impurity which is introduced in them. The experimental situation is in complete agreement with these considerations [5].

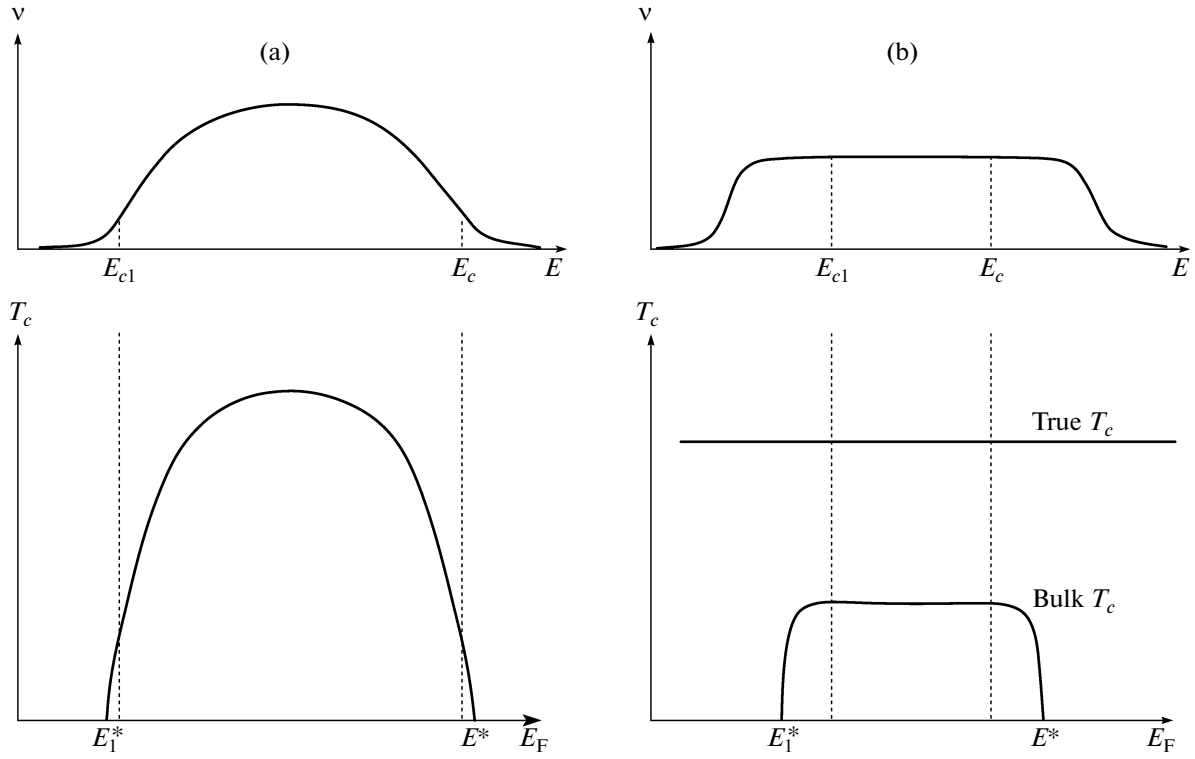

Fig. 3. (a) If disorder is created by weak impurities, self-averaging of the order parameter is not violated; T_c strongly falls near the mobility edge due to decrease of the density of states. (b) In case of strong disorder, the true T_c is determined by localization of the order parameter at rare resonant impurities, while the "bulk T_c" obeys to the "rectangular" dependence.

In case of strong disorder (Fig. 3b), the mobility edge can be located in the region of the practically uniform density of states,[3] so the T_c value given by the Anderson theorem does not fall in approaching the localization threshold. In fact, the true T_c appears to be much larger and corresponds to localization of the order parameter at the small number of the "resonant" impurities, which produce the quasi-local states near the Fermi level. In accordance with papers [9, 10], T_c depends on the interaction constant g in the power-law manner, but contrary to them, it has no essential dependence on the Fermi level position. It removes an illusion that localization "helps" superconductivity. In the vicinity of the true T_c, observation of superconductivity is practically impossible due to a small fraction of the Meissner phase and negligible values of the critical current. Superconductivity becomes easily observable when it spreads to the whole volume: it occurs at some effective temperature, which we refer as the "bulk T_c"; it can be defined theoretically as a transition temperature of the system with removed "resonant" impurities. Such "bulk T_c" corresponds qualitatively (but not quantitatively) to the Anderson theorem and the Bulaevskii–Sadovskii picture is confirmed at this level. In such a case, T_c obeys the "rectangular" dependence (Fig. 3b), which exponentially weakly deviates from the horizontal line near the mobility edge E_c, and exponentially weakly deviates from the vertical line near the endpoint E^* of superconductivity. Such situation is typical for high T_c superconductors, where coexistence of localization and superconductivity is easily observable [5].

The estimate for the true T_c

$$T_c \sim ga^{-d} \sim \lambda_0 J \tag{2}$$

(a is the lattice spacing, J is a bandwidth, and d is the space dimension) gives an impression that the "room" superconductivity is a widespread phenomenon. In fact, the growth of T_c with the increase of λ_0 is bounded by a quantity ω_0/π, where ω_0 is a cut-off frequency. For the phonon mechanism, such upper bound corresponds to the values already attained in high T_c superconductors, and their further increase requires the use of higher frequency Bose excitations. In addition, the observation of true T_c is probably possible only with the use of the scanning tunnel or squid microscopy [16].

It should be stressed that Eq. (2) is a result of the mean field theory. The corresponding solution for the order parameter shows existence of the certain uni-

[3] According to results by Zharekeshev [15] for the strongly disordered Anderson model there is a wide plateau for the density of states in the center of band.

form contribution with abrupt peaks near the rare resonant impurities (with concentration T_c/J). The order parameter can be considered as positive (see Section 2) and so its phase is constant in the whole volume. In the fluctuational theory, the modulus of the order parameter remains practically unchanged, while the essential phase fluctuations arise. If the uniform contribution is neglected, then the system is divided into practically independent superconducting "drops", whose phases are fluctuating freely and destroy the macroscopical coherence of the superconducting state. If the uniform contribution is taken into account, the Josephson coupling between drops arises and their phases become correlated. The accurate fluctuational analysis of such a system is non-trivial, but the general character of results is the same as for the granular superconductors [17]. If the ratio T_c/J is not too small, then the resonant impurities are close to each other and their Josephson interaction is strong enough for stabilization of the mean-field solution at practically the same T_c value (in this sense it can be qualified as "true"); if a concentration of the resonant impurities appears to be small, then T_c is suppressed by fluctuations to the value somewhat greater than the "bulk T_c" (Section 7).

According to the results of [9, 10][4]

$$T_c \sim g^{1/\gamma}, \quad f \sim (T_c/J)^{\gamma}, \tag{3}$$

where f is a portion of volume occupied by superconductivity, and parameter $\gamma = 0.57$ is related with a fractal dimensionality of wave functions. We do not deny the existence of the order parameter configurations, leading to results of type (3) (Section 3), but Eq. (2) corresponds to the higher value of T_c; the corresponding configuration of the order parameter is determined by the rare peaks near the resonant impurities, occurring at the atomic scale and occupying a portion of volume $f \sim T_c/J$. If superconductivity is considered as a variational problem, then it is possible to say that our trial function is more successful than one in [9, 10]. Formally, our results correspond to Eq. (3) with $\gamma = 1$ and do not contain any information on multifractality; hence, there are no grounds to say on "fractal superconductivity" [10] near the localization threshold.

[4] In the recent paper by Burmistrov et al [18] the results analogous to [9, 10] are obtained in the Finkelstein renormalization group approach [19]. However, these papers are essentially different both in the initial assumptions and in the discussed physical mechanism, so one cannot say that one paper confirms another. The authors of [9, 10] tried to advance beyond the assumption on self-averaging, while a fixed value of the interaction constant is accepted; contrary, [18] takes into account a disorder dependence of the interaction constant, while a self-averaging property is taken for granted. By the latter reason, the present results cannot be reproduced in [18], whereas the considered there effect is more weak.

2. ANDERSON THEOREM AND INEQUALITIES FOR T_c

A basis for description of the spatially inhomogeneous superconductivity is given by the Gor'kov equation for the order parameter $\Delta(r)$

$$\Delta(r) = \int K(r, r')\Delta(r')d^d r' \tag{4}$$

with the kernel $K(r, r')$ in representation of exact one-particle eigenstates $\varphi_s(r)$

$$K(r, r') = gT\sum_{\omega}\sum_{s, s'} \frac{\varphi_s^*(r)\varphi_s(r')\varphi_{s'}^*(r)\varphi_{s'}(r')}{(\epsilon_s - i\omega)(\epsilon_{s'} + i\omega)}, \tag{5}$$

where ϵ_s are eigen energies (counted from the Fermi level), and summation occurs over the Matsubara frequencies $\omega_n = \pi T(2n + 1)$ with integer n. Following de Gennes [20], we use the frequency cut-off $|\omega| < \omega_0$, which corresponds to the electron interaction

$$V(r, r'; \omega) = -g\theta(\omega_0 - |\omega|)\delta(r - r'), \tag{6}$$

which is strictly local and can be specified independently of one-particle eigenstates (in contrast to the momentum cut-off in the original BCS formulation, where interaction is defined by the matrix elements over plane waves). In the absence of magnetic effects, eigenstates $\varphi_s(r)$ can be taken real and their orthogonality leads to the sum rule [20]

$$\int K(r, r')d^d r' = g\nu_F(r)\ln\frac{1.14\omega_0}{T}, \tag{7}$$

where $\nu_F(r) \equiv \nu(0, r)$ is the local density of states

$$\nu(\epsilon, r) = \sum_s |\varphi_s(r)|^2\delta(\epsilon - \epsilon_s) \tag{8}$$

at the Fermi level. It is accepted in derivation of (7) that $\nu(\epsilon, r)$ is a slow function of ϵ on the scale of T_c; generally $\nu_F(r)$ should be understood as a local density of states smoothed at the scale of T_c.

The Anderson theorem follows from Eq. (4) under assumption of a self-averaging order parameter, when $\Delta(r)$ and $K(r, r')$ can be independently averaged over disorder. Since $\langle\Delta(r)\rangle$ does not depend on r due to the spatial uniformity in average, the use of the sum rule (7) gives

$$\langle\Delta\rangle = g\langle\nu_F\rangle\ln\frac{1.14\omega_0}{T}\langle\Delta\rangle, \tag{9}$$

and T_c is given by the BCS formula, which contains the average density of states $\langle\nu_F\rangle$. The latter does not change at the Anderson transition point, suggesting the analogous behavior for T_c. More detailed information can be obtained, if Eq. (4) is averaged over variable r

$$\langle\Delta\rangle = g\ln\frac{1.14\omega_0}{T}\langle\nu_F(r)\Delta(r)\rangle. \tag{10}$$

The function $\Delta(r)$ can be considered as positive,[5] and one has

$$\nu_{\min}\langle\Delta\rangle < \langle\nu_F(r)\Delta(r)\rangle < \nu_{\max}\langle\Delta\rangle, \quad (11)$$

where $\nu_{\min}$ and $\nu_{\max}$ are the minimal and maximal values of $\nu_F(r)$. It gives inequalities for T_c

$$1.14\omega_0\exp(-1/g\nu_{\min}) < T_c < 1.14\omega_0\exp(-1/g\nu_{\max}), \quad (12)$$

which can be also obtained from the known theorems of the matrix theory [13, Section 2]. According to Eq. (12), the power law dependence of T_c on the coupling constant g [9, 10] is impossible, if $\nu_F(r)$ has an upper bound $\nu_{\max}$.

Near the Anderson transition, there are systematic reasons for growth of the $\nu_F(r)$ fluctuations [3, 4]. As noted in [5], the correlator $\langle\nu(E+\omega, r)\nu(E, r')\rangle$ at $r = r'$ coincides with the Berezinskii–Gor'kov spectral density [22], which is determined by the diffusion pole with the observable diffusion coefficient $D(\omega, q)$ [23]:

$$\langle\nu(E+\omega, r)\nu(E, r)\rangle \sim \mathrm{Re}\int\frac{d^dq}{-i\omega + D(\omega, q)q^2}. \quad (13)$$

In the metallic phase, the static diffusion constant $D(0, q)$ is real, so $\langle\nu_F(r)^2\rangle$ diverges at the transition point as D^{-1}. In the dielectric phase, the analogous estimate can be obtained from the self-consistent theory of localization [24] by iteration of [23, Eq. (112)]

$$D(\omega, q) = (-i\omega)d(q) + \omega^2 d_1(q), \quad d(q) \sim \xi^2, \quad d_1(q) \sim \xi^4|\tau|^{-1} \quad (14)$$

(τ is a distance to the critical point), so $\langle\nu_F(r)^2\rangle \sim |\tau|^{-1}$ and fluctuations grow symmetrically on two sides of the transition.[6] Estimations of the correlator (13) at the critical point based on multifractality of wave functions [10] suggest the dependence $\omega^{-\gamma}$ for $\omega \longrightarrow 0$; if divergency is cut off at the scale T_c, then $\nu_{\max} \sim \nu_0(J/T_c)^{\gamma/2}$ and the maximum value $T_c \sim g^{2/\gamma}$ allowed by Eq. (12) is in a qualitative agreement with [9, 10]. Consequently, if the upper bound for T_c is realized in Eq. (12), then it reaches the maximum value at the localization threshold depending on g in the power law manner.

However, the distribution of quantities $|\varphi_s(r)|^2$ has the power law tails [10] and Eq. (13) determines neither the typical, nor the maximal value of $\nu_F(r)$. In fact, the given estimate for T_c is not reached for weak disorder and is exceeded for strong disorder. Formally, the approach of [10] is questionable due to replacement of matrix elements $M_{ijkl} = \int d^dr\varphi_i(r)\varphi_j(r)\varphi_k(r)\varphi_l(r)$ by their mean values with averaging independently of the order parameter.

More efficient approach is based on the study of effects from individual impurities, since it allows to work with specific realizations of the random potential and contains no problems of averaging. Introducing one impurity after another, one can easily be convinced (Section 4), that unbounded values of $\nu_F(r)$ can arise only from existence of quasi-local states (Fig. 2). The problem of quasi-local states has a general character. Indeed, one can imagine such fluctuation of the random potential, that a finite region of space is isolated from its environment by the high barrier; the corresponded discrete levels can have a very weak broadening and, appearing close to the Fermi level, can lead to unbounded values of $\nu_F(r)$. Such problems are discussed in the next section.

3. RESONANCES AT QUASI-DISCRETE LEVELS

Suppose that a system has a discrete spectrum and only one state is close to the Fermi level; then we can retain only one term in the sum over s, s' in (5):

$$K(r, r') = gT\sum_\omega\frac{\varphi_0^2(r)\varphi_0^2(r')}{\epsilon_0^2+\omega^2} \equiv gA(T)\varphi_0^2(r)\varphi_0^2(r'). \quad (15)$$

Then Eq. (4) gives

$$\Delta(r) = X\varphi_0^2(r), \quad X = gA(T)\int\varphi_0^2(r')\Delta(r')d^dr' \quad (16)$$

and self-consistency of these equations determines T_c:

$$1 = gA(T)I_4, \quad I_4 = \int\varphi_0^4(r)d^dr. \quad (17)$$

Calculation of $A(T)$ is possible without the cut-off frequency taken into account, since the sum converges at large ω:

$$A(T) = T\sum_\omega\frac{1}{\epsilon_0^2+\omega^2} = \frac{1}{2\epsilon_0}\tanh\frac{\epsilon_0}{2T}. \quad (18)$$

For the exact resonance ($\epsilon_0 = 0$) we have $A(T) = 1/4T$, so

$$T_c = gI_4/4, \quad (19)$$

and T_c has a power law dependence on the interaction constant g. In the general case (see Fig. 4a)

$$T_c = \frac{\epsilon_0}{\ln(\epsilon_c+\epsilon_0) - \ln(\epsilon_c-\epsilon_0)}, \quad \epsilon_c = \frac{gI_4}{2} \quad (20)$$

[5] For real $\varphi_s(r)$, the kernel $K(r, r')$ is positive, since it can be written as $gT\sum_\omega|G_\omega(r, r')|^2$ (see Eq. (5)). The Cooper instability corresponds to the minimal characteristic number (or maximal eigenvalue) and the nodeless eigenfunction (the Entch theorem) [21].

[6] According to the self-consistent theory, $D \sim \tau$ in the metallic phase [24].

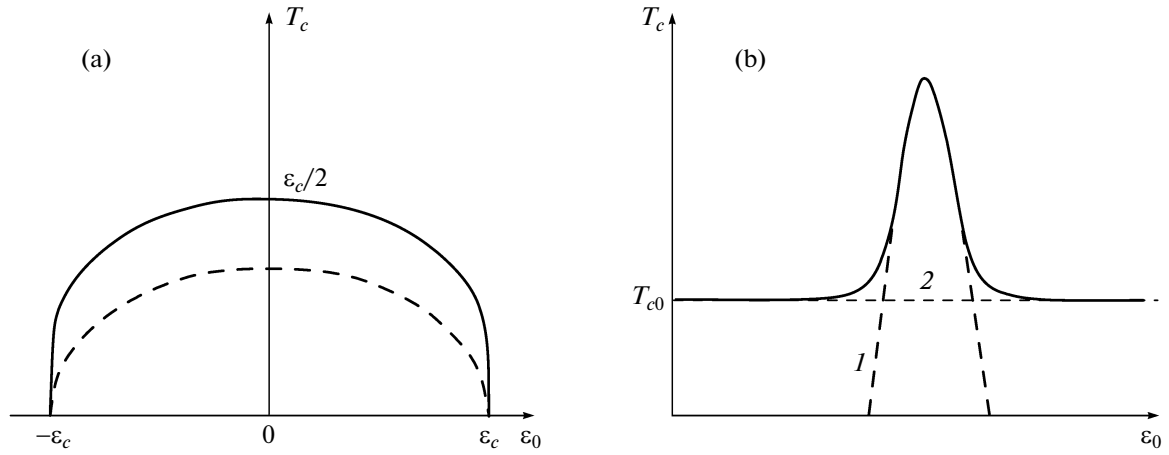

Fig. 4. (a) T_c of the one-level system as a function of the level position ϵ_0 in the absence of attenuation (solid line); attenuation γ produces a shift of the curve by the quantity $\sim\gamma$ (dashed line). (b) The same, for a situation when the quasi-discrete level lies in the background of the continuous spectrum. *1*—Localized superconductivity, *2*—bulk T_c.

and a solution exists under condition

$$|\epsilon_0| < \epsilon_c. \tag{21}$$

At first glance, the considered regime is destroyed due to fluctuations[7] or coupling with the continuous spectrum; in fact, it is not so (see below) and the main problem consists in the possibility to match the discrete level with the Fermi energy.

Indeed, let the system has a finite size L, while its eigenstates are extended. Then the Fermi energy is located between two discrete levels,[8] and ϵ_0 is determined by the average level spacing $J(L/a)^{-d}$; estimating $I_4 \sim L^{-d}$ from the normalization condition, we see that

$$\epsilon_0 \sim J(L/a)^{-d}, \quad \epsilon_c \sim gL^{-d} \tag{22}$$

and condition (21) cannot be fulfilled in the weak coupling regime, which is the only allowable in the BCS scheme.

Let us couple the given system with a reservoir, and try to match the chemical potential of the latter with the discrete level of the system. However, nothing good will occur from it: the local Fermi level of the system is still arranged between two discrete levels and it tends to equalize with the Fermi energy in the reservoir. The real flow of electrons is impossible due to elecroneutrality, and the problem will be solved by a minimal deformation: a double layer will arise between the reservoir and the system, and it will equate the Fermi levels.

By the same reason, the situation cannot be improved due to localization of states. At first glance, in this case $\epsilon_0 \sim J(L/a)^{-d}$, $\epsilon_c \sim g\xi^{-d}$ (ξ is the localization radius of $\varphi_0(r)$), so condition (21) reduces to $g \gtrsim Ja^d(L/\xi)^{-d}$ and can be fulfilled at sufficiently large L. In fact, blocks of size ξ are quasi-independent and each of them has its own local Fermi level; these levels equalize due to double layers between blocks, and the given estimates are valid only for $L \sim \xi$. In fact, the above arguments clarify the mechanism for the Coulomb gap [25].

It looks that the only possibility to avoid the given arguments is to take the size L of the atomic order. Indeed, at such a scale: (a) the notion of the Fermi level becomes senseless; (b) electroneutrality can be violated; (c) a size of the double layer is comparable with L. It means that the strong violations of the Anderson theorem can be exhaustively analyzed by consideration of the one-impurity problem (Section 4).

Already at this stage it is possible to establish the relation with results of [9, 10]. In the considered there strictly one-electron picture, the discrete system of levels fluctuates freely relative to the Fermi energy, so resonances are possible at any length scale L. Then all principal statements of [9, 10] are reproduced: T_c has a power law behavior as a function of g and does not depend on the cut-off frequency ω_0, while the order parameter $\Delta(r)$ follows the form of the wave function (see (16)) and will have multifractal properties simultaneously with multifractality of the latter.[9] However, this picture is completely destroyed, if electroneutrality is taken into account, since resonances at large scales become impossible. In fact, large scale fluctuations are insignificant even in a strictly one particle picture: a value of T_c for an exact resonance, $T_c \sim gL^{-d}$ (see (17), (19)), is greater for small scales.

[7] If eigenstate $\varphi(r)$ is localized, then according to (20) a superconducting transition takes place in a finite system; of course, such conclusion is an artifact of the mean field theory and in fact the transition is destroyed by fluctuations.

[8] For a discussion of the parity effect see Footnote 19.

[9] We have no doubt that papers [9, 10] implicitly dealt with the same effect, but the improper averaging procedure led to a domination of large length scales.

Generally, the considered regime is not destroyed in the presence of the continuous spectrum. In this case, the level ϵ_0 acquires the finite decay γ, which can be taken into account by replacement

$$\epsilon_0 \pm i\omega \longrightarrow \epsilon_0 + i\omega \pm i\gamma\,\mathrm{sgn}\,\omega, \tag{23}$$

so

$$A(T) = T\sum_{\omega}\frac{1}{\epsilon_0^2 + (|\omega| + \gamma)^2} \approx \frac{1}{\pi\epsilon_0}\arctan\frac{\epsilon_0}{\gamma + bT}, \tag{24}$$

where we have estimated the sum by the integral, introducing the cut-off $|\omega| > bT$ (for the choice $b = 4/\pi$ such estimate practically coincides with the exact result (18) for $\gamma = 0$). The finiteness of γ leads qualitatively to the shift of the curve in Fig. 4a by a quantity $\sim\gamma$, so a solution survives for $\gamma \lesssim \epsilon_c$.

For finite ω_0 one obtains instead (24)

$$A(T) \approx \frac{1}{\pi\epsilon_0}\arctan\frac{\epsilon_0(\omega_0 - bT)}{\epsilon_0^2 + (\omega_0 + \gamma)(\gamma + bT)}, \tag{24'}$$

and can be easily convinced that finiteness of ω_0 is irrelevant under condition $\omega_0 \gg \epsilon_c$. In the opposite case $\omega_0 \ll \epsilon_c$ the allowed values of ϵ_0 and γ have an order $(\epsilon_c\omega_0)^{1/2}$, while the maximal critical temperature T_c is of the order ω_0; in fact, restriction $T_c < \omega_0/\pi$ is evident, since for $T > \omega_0/\pi$ the sum over ω contains no terms.

To investigate the effect of the continuous spectrum on the order parameter, one can use the following approximation for the kernel $K(r, r')$

$$K(r, r') = K_0(r - r') + gA(T)\varphi_0^2(r)\varphi_0^2(r'), \tag{25}$$

which ignores the backward influence of the discrete level on the continuous spectrum. According to [11, 12], such approximation provides qualitatively correct description and can be justified in certain limiting cases.[10]

Having in mind a consideration of periodical configurations, we solve Eq. (4) with the kernel (25) for a finite system of size L with the periodical boundary conditions. We accept $L \ll \xi_0\tau^{-1/2}$, where $\tau = (T - T_{c0})/T_{c0}$, ξ_0 is the coherence length, and T_{c0} is a transition temperature, corresponding to the continuous spectrum.[11] After the Fourier transform one has

$$\Delta_q = gA(T)X\frac{\langle\varphi_0^2\rangle_q}{1 - K_0(q)}, \qquad X = L^{-d}\sum_q \langle\varphi_0^2\rangle_{-q}\Delta_q \tag{26}$$

and self-consistency of two expressions leads to

$$1 = gA(T)\left[L^{-d}\sum_q \langle\varphi_0^2\rangle_q\langle\varphi_0^2\rangle_{-q} + L^{-d}\sum_q \frac{K_0(q)}{1 - K_0(q)}\langle\varphi_0^2\rangle_q\langle\varphi_0^2\rangle_{-q}\right]. \tag{27}$$

Using expansion in q^2

$$1 - K_0(q) = \lambda_0\tau + \frac{1}{2}\lambda_0\xi_0^2q^2 + \ldots, \tag{28}$$

it is easy to see that the main contribution to the second sum in (27) occurs from small q, and the single term with $q = 0$ is sufficient for $L \ll \xi_0\tau^{-1/2}$. Then Eq. (27) accepts a form

$$1 = gA(T)\left[I_4 + \frac{1}{\lambda_0 L^d\tau}\right], \tag{29}$$

and the analogous approximations in (26) give

$$\Delta(r) = \mathrm{const}\left[\varphi_0^2(r) + \frac{1}{\lambda_0 L^d\tau}\right]. \tag{30}$$

If $T_{c0} \ll \epsilon_c$, then dependence of T_c on ϵ_0 has a form shown in Fig. 4b. In the zero approximation there are two independent systems, the quasi-local one with the transition temperature (20) (if attenuation γ is small) and the continuous one characterizing by T_{c0}, while T_c of the composed system is given by the maximal of two values. Interrelation of two systems reduces to smoothing of dependence $T_c(\epsilon_0)$ at the scale $T_{c0}(a/L)^{d/2}$, if $\varphi(r)$ is localized at the atomic scale a.

It is clear from (30) that the order parameter $\Delta(r)$ is practically constant for small τ and localized at the scale a for large τ. Crossover from one regime to another is very abrupt, and one can say on the "Anderson transition" for superconducting electrons. We see that the localized regime survives in the presence of the continuous spectrum, if the corresponding T_c exceeds T_{c0}. In fact, existence of the continuous spec-

[10] In Section 4. we consider the one-impurity problem with the backward influence on the continuous spectrum.

[11] Appearance of the characteristic scale $\xi(T) = \xi_0\tau^{-1/2}$ was discussed previously [11] for the case of plane defects. If $L \gg \xi(T)$, then individual defects becomes practically independent and the order parameter is localized near them on the scale $\xi(T)$. In the opposite case $L \ll \xi(T)$, the order parameter is practically constant in the space between defects. Below (Section 5) we consider configurations with small concentration ($\sim T_c/E_F$) of the resonant impurities, so the distance between them $a(E_F/T_c)^{1/3}$ is less than $\xi_0 \sim a(E_F/T_c)$.

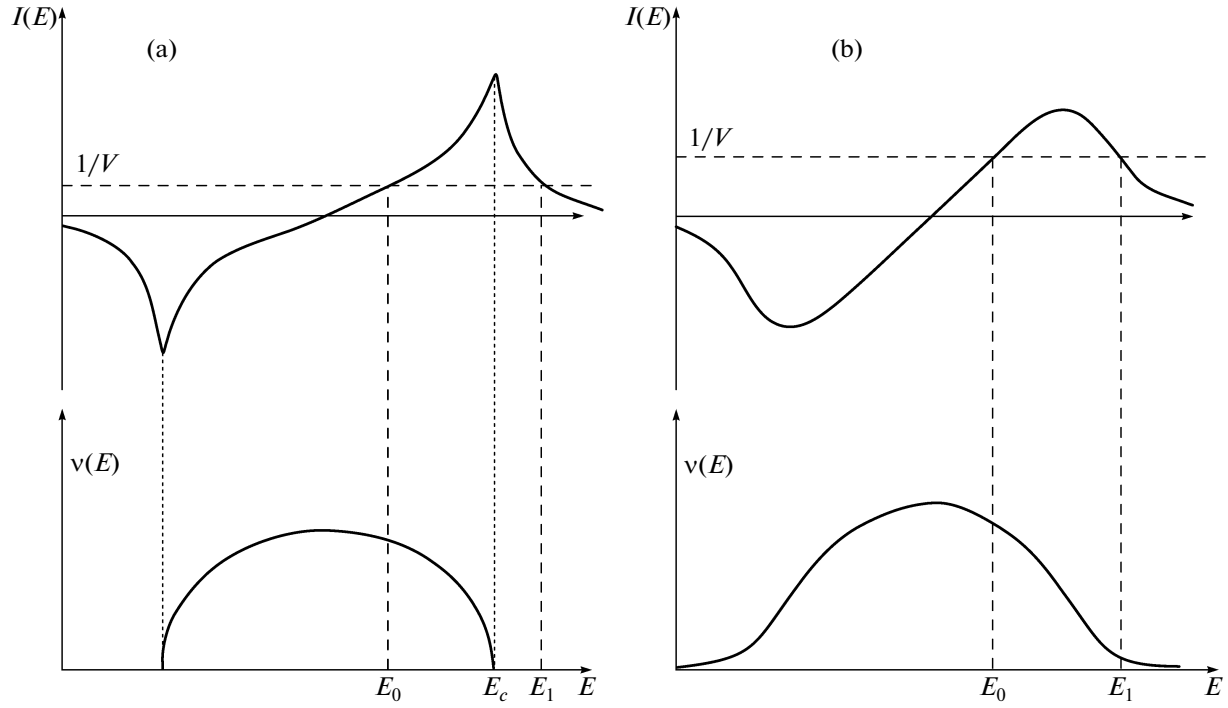

Fig. 5. (a) If the impurity $V\delta_{nn_0}$ is inserted in an ideal lattice, equation $1 = VI(E)$ has two roots for large V, E_1, and E_0, which correspond to the local and quasi-local levels. (b) If the same impurity is inserted in the disordered lattice, both solutions correspond to the quasi-local levels.

trum has a stabilizing effect on the localized superconductivity, since the order parameter takes non-zero values in the whole volume.

4. ONE-IMPURITY PROBLEM

If $G^0_{nn'}$ is the Green function of an ideal lattice, $V_n = V\delta_{nn_0}$ is an impurity potential, then the Green function $G_{nn'}$ of the perturbed system is determined by the Dyson equation [26]:

$$G_{nn'} = G^0_{nn'} + G^0_{nn_0} V G_{n_0 n'}. \tag{31}$$

Setting $n = n_0$, one has the closed equation for $G_{n_0 n'}$, whose solution is substituted into (31)

$$G_{nn'} = G^0_{nn'} + G^0_{nn_0}\mathcal{T}G^0_{n_0 n'},$$
$$\mathcal{T} = \frac{V}{1 - VG^0_{n_0 n_0}}, \tag{32}$$

where the scattering $\mathcal{T}$-matrix reduces to a constant in the given case. For an ideal lattice G^0_{nn} does not depend on n,

$$G^0_{nn} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{E - \epsilon_k + i0}$$
$$= \int \frac{\nu_0(\epsilon) d\epsilon}{E - \epsilon + i0} \equiv I(E) - i\pi\nu_0(E), \tag{33}$$

and so $\mathcal{T}$-matrix has no n_0 dependence. Condition $1 - VI(E) = 0$ corresponds to existence of the local (if $\nu_0(E) = 0$) or quasi-local (if $\nu_0(E) \neq 0$) level [26, 27] (Fig. 5). The local density at site n_0

$$\nu(E, n_0) = \frac{\nu_0(E)}{[1 - VI(E)]^2 + [\pi V \nu_0(E)]^2} \tag{34}$$

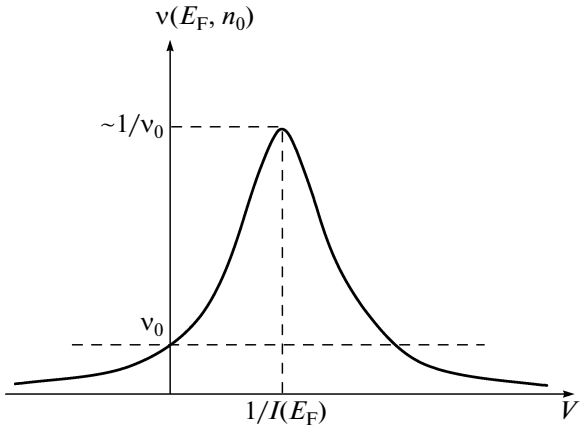

Fig. 6. The local density of states at the point n_0 as a function of the impurity potential V.

has an abrupt maximum near the resonance $1 - VI(\epsilon_0) = 0$ (Fig. 6) with a value in it

$$[\nu(E, n_0)]_{\text{res}} = I(\epsilon_0)^2/\pi^2\nu_0(\epsilon_0) \tag{35}$$

which grows unboundedly near the initial band edge (where $\nu_0(\epsilon_0) \longrightarrow 0$). In the vicinity of the band edge, a calculation of $G^0_{nn'}$ is possible in the continual approximation and gives at $d = 3$ (for $|r - r'| \gtrsim a$):

$$G^0_{nn'} \longrightarrow G^0(r - r') = -\pi\nu_0\frac{e^{ik_F|r-r'|}}{k_F|r-r'|}. \tag{36}$$

Deviation of $\nu(E, n)$ from $\nu_0(E)$ is maximal for $n = n_0$ and tends to zero for $|n - n_0| \longrightarrow \infty$.

The Matsubara representation for the Green functions is obtained from (32) by replacement $E \longrightarrow \epsilon_F + i\omega$, where $\epsilon_F = 0$ for the corresponding choice of the energy origin. Setting $n_0 = 0$, one has for the kernel in (4)

$$\begin{aligned} K(r, r') = gT\sum_\omega |G_\omega(r, r')|^2 = gT\sum_\omega \left|G^0_\omega(r - r')\right|^2 \\ + gT\sum_\omega G^0_\omega(r)\mathcal{T}_\omega G^0_\omega(r')G^0_{-\omega}(r - r') \\ + gT\sum_\omega G^0_{-\omega}(r)\mathcal{T}_{-\omega} G^0_{-\omega}(r')G^0_\omega(r - r') \\ + gT\sum_\omega \left|G^0_\omega(r)\right|^2|\mathcal{T}_\omega|^2\left|G^0_\omega(r')\right|^2 \\ \equiv K_0(r - r') + K_1(r, r'). \end{aligned} \tag{37}$$

Solution of the Gor'kov equation with the kernel (37) is sought in the form

$$\Delta(r) = \Delta_0 + \Delta_1(r), \tag{38}$$

where $\Delta_1(r)$ is localized near $r = 0$. Substituting in (4) and using the sum rule (7), one has

$$\begin{aligned} \Delta(r) = \int K_0(r - r')\Delta(r')d^3r' \\ + g\nu_1(r)\ln\frac{1.14\omega_0}{T}\Delta_0 + F(r), \end{aligned} \tag{39}$$

where $\nu_1(r)$ is deviation of the local density of states from ν_0 and

$$F(r) = \int K_1(r, r')\Delta_1(r')d^3r'. \tag{40}$$

Consideration of the isolated impurities is not actual (see Footnote 7), so we accept their periodical arrangement and solve the Gor'kov equation for a finite system of size L with periodical boundary conditions for $L \ll \xi_0\tau^{-1/2}$. Resolving (39) for $\Delta(r)$ by the Fourier transform and simplifying the result analogously to (27), it is possible to separate the uniform term corresponding to Δ_0, while the rest is attributed to $\Delta_1(r)$ ($\langle \ldots \rangle_0$ is the zero Fourier component):

$$\begin{aligned} \Delta_0 = \frac{K_0(0)}{\lambda_0 L^3\tau}\left[\Delta_0\ln\frac{1.14\omega_0}{T}g\langle\nu_1\rangle_0 + \langle F\rangle_0\right], \\ \Delta_1(r) = \Delta_0\ln\frac{1.14\omega_0}{T}g\nu_1(r) + F(r). \end{aligned} \tag{41}$$

Using the explicit expression for $F(r)$ and setting in the integrals[12]

$$\Delta_1(r')G^0_\omega(r - r') \approx \Delta_1(r')G^0_\omega(r), \tag{42}$$

one can transform (41) to the form

$$\Delta_0 = \frac{K_0(0)}{\lambda_0 L^3\tau}\ln\frac{1.14\omega_0}{T}[g\langle\nu_1\rangle_0\Delta_0 + g\langle\nu_1\Delta_1\rangle_0], \tag{43a}$$

$$\begin{aligned} \Delta_1(r) = \Delta_0\ln\frac{1.14\omega_0}{T}g\nu_1(r) \\ + gT\sum_\omega Z_\omega\left|G^0_\omega(r)\right|^2, \end{aligned} \tag{43b}$$

where

$$\begin{aligned} Z_\omega = \mathcal{T}_\omega Y_\omega + \mathcal{T}_{-\omega}Y_{-\omega} + |\mathcal{T}_\omega|^2 X_\omega, \\ X_\omega = \int\Delta_1(r)\left|G^0_\omega(r)\right|^2 d^3r, \\ Y_\omega = \int\Delta_1(r)G^0_\omega(r)d^3r. \end{aligned} \tag{44}$$

Substituting $\Delta_1(r)$ from (43b) into expressions (44), and estimating arising integrals

$$\begin{aligned} g\int\nu_1(r)\left|G^0_\omega(r)\right|^2 d^3r \equiv \lambda_{01}, \\ g\int\left|G^0_\omega(r)\right|^2\left|G^0_{\omega'}(r)\right|^2 d^3r \equiv \lambda_{11}, \\ g\int\nu_1(r)G^0_\omega(r)d^3r \equiv \lambda'_{02} + i\lambda''_{02}\operatorname{sgn}\omega, \\ g\int G^0_\omega(r)\left|G^0_{\omega'}(r)\right|^2 d^3r \equiv \lambda'_{12} + i\lambda''_{12}\operatorname{sgn}\omega, \end{aligned} \tag{45}$$

with the use of expressions for $G^0_\omega(r)$ and $\nu_1(r)$ (where the real and imaginary parts are denoted by a prime and two primes)

$$\begin{aligned} G^0_\omega(r) = -\frac{\pi\nu_0}{k_F r}\exp\left\{-\frac{|\omega|}{v_F}r + ik_F r\operatorname{sgn}\omega\right\}, \\ \nu_1(r) = -\pi\nu_0^2\frac{\mathcal{T}''\cos 2k_F r + \mathcal{T}'\sin 2k_F r}{(k_F r)^2}, \end{aligned} \tag{46}$$

it is easy to see that the integrals converge already for $\omega, \omega' = 0$, so parameters λ_{01}, λ_{11}, etc. can be considered as constant; it allows to write (44) in the form

[12] This approximation is not quite rigorous, but in fact it is used only for estimates: the corresponding terms characterized by parameters λ_{02} and λ_{12} have no significance both far from the resonance, and in its vicinity (see Appendix).

$$X_\omega = X, \quad Y_\omega = Y' + iY''\operatorname{sgn}\omega,$$
$$Z_\omega = 2\mathcal{T}'_\omega Y' - 2\left|\mathcal{T}''_\omega\right| Y'' + \left|\mathcal{T}_\omega\right|^2 X. \tag{47}$$

The region remote from the resonance. The natural scale for the energy dependence of $\mathcal{T}$-matrix is given by the bandwidth J, so $\mathcal{T}'_\omega$ and $\left|\mathcal{T}''_\omega\right|$ can be considered as independent of ω anywhere, excepting the vicinity of the resonance (see below). Then Z_ω is also independent of ω, and substitution of $\Delta_1(r)$ from (43b) into (44) leads to the linear system of equations for Δ_0 and Z (see Appendix), whose solubility condition gives

$$\frac{\delta T_c}{T_{c0}} = \frac{1}{\lambda_0 L^3}\int d^3r \frac{\nu_0\nu_1(r) + \nu_1(r)^2}{\nu_0^2}. \tag{48}$$

Equation (48) is a natural generalization of the result (1): the first term in the numerator corresponds to the Anderson theorem, while the second determines corrections to it. A configuration of the order parameter shows that (48) corresponds to the delocalized regime.

For weak impurities ($|V| \ll J$) one has the estimates

$$\mathcal{T}' \approx \mathcal{T} \sim Va^3, \quad \mathcal{T}'' \sim Va^3(V/J)(k_F a), \tag{49}$$

and

$$\nu_0\langle\nu_1\rangle_0 \sim a^3\nu_0^2(V/J)(k_F a)^{-2},$$
$$\langle\nu_1^2\rangle_0 \sim a^3\nu_0^2(V^2/J^2)(k_F a)^{-1}, \tag{50}$$

so the Anderson term is leading both in parameter V/J and in parameter $(k_F a)^{-1}$. We accepted here $k_F a \ll 1$, having in mind a situation near the band edge, while estimates for the band center follow at $k_F a \sim 1$.

The delocalized regime retains in the case when the resonance condition $1 \approx VI(E_F)$ is formally fulfilled, but the density of states $\nu(E_F)$ is sufficiently large to provide a strong attenuation of the quasi-local state. In this situation

$$\mathcal{T}' \sim Ja^3\frac{J\epsilon_0}{\gamma^2}, \quad \mathcal{T}'' \sim Ja^3\frac{J}{\gamma} \tag{51}$$

and one has under condition $\gamma \gtrsim \epsilon_0$ (where ϵ_0 and γ are defined in Eq. (54))

$$\nu_0\langle\nu_1\rangle_0 \sim \langle\nu_1^2\rangle_0 \sim a^3\nu_0^2(k_F a)^{-2}, \tag{52}$$

i.e., the Anderson term has the same order, as a correction to it.

Vicinity of the resonance. If ϵ_0 is a root of equation $1 = VI(\epsilon)$, then in the vicinity of it

$$1 - VI(\epsilon) = \frac{\epsilon - \epsilon_0}{E_0}, \quad E_0 \sim J, \tag{53}$$

and hence

$$\mathcal{T} = \frac{VE_0}{\epsilon - \epsilon_0 + i\gamma}, \quad \gamma = \pi VE_0\nu(\epsilon_0). \tag{54}$$

In the Matsubara representation one has

$$\mathcal{T}_\omega = \frac{VE_0}{i\omega - \epsilon_0 + i\gamma\operatorname{sgn}\omega} \equiv \mathcal{T}'_\omega - i\left|\mathcal{T}''_\omega\right|\operatorname{sgn}\omega, \tag{55}$$

where

$$\mathcal{T}'_\omega = -VE_0\frac{\epsilon_0}{\epsilon_0^2 + (|\omega| + \gamma)^2},$$
$$\left|\mathcal{T}''_\omega\right| = VE_0\frac{|\omega| + \gamma}{\epsilon_0^2 + (|\omega| + \gamma)^2}, \tag{56}$$

so $\mathcal{T}$-matrix can be considered as independent of ω under condition

$$|\omega| \ll \gamma \ \text{ or } \ k_F a \gg \omega/J, \tag{57}$$

i.e., not very close to the initial band edge. If this condition is not fulfilled,[13] then the ω dependence is essential for the quantities $\mathcal{T}'_\omega$ and $\left|\mathcal{T}''_\omega\right|$, and hence for Z_ω. In fact, only one combination is relevant,

$$S = T\sum_\omega Z_\omega, \tag{58}$$

and substitution of $\Delta_1(r)$ from (43b) into (44) allows to express Z_ω through Δ_0 and S; substituting this expression for Z_ω into (58) and (43), one comes to the linear system of equations for Δ_0 and S; its solubility condition with only leading terms retained (see Appendix) reduces to

$$\left(\tau - \frac{\nu_0\langle\nu_1\rangle_0 + \langle\nu_1^2\rangle_0}{\lambda_0 L^3\nu_0^2}\right)$$
$$\times\left(-1 + \lambda_{11}V^2E_0^2A(T)\right) + \tau_c^2 = 0, \tag{59}$$

where $A(T)$ corresponds to expression (24). Equation (59) describes the typical situation related with intersection of terms. The zero of the first bracket corresponds to the delocalized regime (see Eq. (48)), and vanishing of the second bracket corresponds to equation for T_c of the localized superconductivity (compare with (17), (24)), while the term $\tau_c^2 \sim (a/L)^3$ removes the degeneracy of terms in the intersection point (Fig. 7).

5. CONSEQUENCES FOR THE ANDERSON MODEL

Usually localization is studied in the framework of the Anderson model, which is a discrete version of the Schrödinger equation with a random potential: the bare spectrum is a band of width J, while the potential values V_n on the lattice sites are independent random quantities with the distribution $P\{V\}$ of width $\sim W$, which is supposed to be rectangular. To transfer from

[13] In this case, the factor $\exp(-|\omega|\, r/v_F)$ restricts contribution to the integrals (45) by the atomic scale, where expression (46) are inapplicable.

the one-impurity problem to the Anderson model, it is sufficient to accept that a potential V of impurities fluctuates in the interval $(-W, W)$, while their concentration c is gradually increased from small values to values of the order of unity.

The results of Section 4 correspond formally to the periodic arrangement of impurities, but in fact their periodicity is not essential: each impurity arouses only a local deformation of the order parameter and these deformations are independent in case of a small concentration. If $\Delta_1(r)$ and $\nu_1(r)$ correspond to the one-impurity problem, then configurations

$$\Delta(r) = \Delta_0 + \sum_i \Delta_1(r - r_i), \qquad \nu_F(r) = \nu_0 + \sum_i \nu_1(r - r_i) \tag{60}$$

correspond to a situation, when several impurities are arranged in points r_i: it is a consequence of localization of the kernel $K_1(r, r')$ in both variables near the defect position. It is clear from (41) that the amplitude of $\Delta_1(r)$ is proportional to Δ_0, so $\Delta_1(r) = \Delta_0 f(r)$ and substitution of (60) into (10) gives for T_c close to T_{c0}:

$$\frac{\delta T_c}{T_{c0}} = \frac{m}{\lambda_0^2 L^3} g \int d^3 r [\nu_1(r) + \nu_1(r) f(r)]. \tag{61}$$

Here, m is a number of impurities in the volume L^3, and $f(r)$ can be identified as $\nu_1(r)/\nu_0$ from comparison with (48).[14] One can see that effect is proportional to a concentration of impurities, while their arrangement is irrelevant. The Anderson theorem is valid under condition $|f(r)| \ll 1$, which is fulfilled for weak impurities. In case of nonequivalent impurities, the result (61) should be averaged according to distribution $P(V)$.

In the regime of the localized order parameter, each impurity is practically independent of environment and T_c of the system is determined by those of them, which are close to a resonance; if distribution $P(V)$ is continuous and sufficiently wide, then the condition of almost exact resonance is always realized with a certain probability. Therefore, the concentration of the resonant impurities is finite and their quasiperiodic arrangement stabilizes the mean-field solution.

Above considerations completely clarify a situation for small impurity concentrations. Advancement to higher concentrations is simplified by observation that equations (31), (32) never used a fact that $G^0_{nn'}$ corresponds to an ideal lattice; the same equations describe

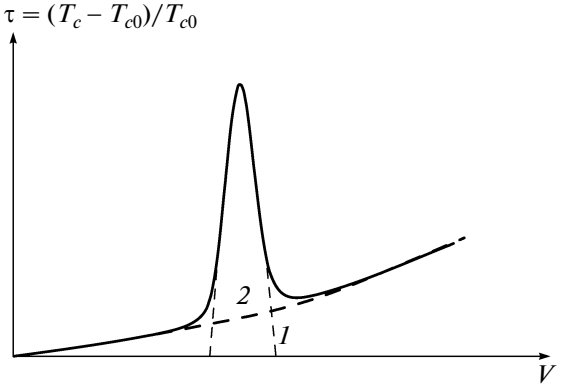

Fig. 7. Dependence of T_c on the impurity potential V for a small impurity concentration. (*1*) Localized superconductivity; (*2*) delocalized regime.

insertion of an additional impurity in the disordered superconductor. Noticing that

$$G^0_{n_0 n_0} = \sum_s \frac{|\varphi_s(n_0)|^2}{E - \epsilon_s + i0} = \int d\epsilon \frac{\nu(\epsilon, n_0)}{E - \epsilon + i0}$$

and replacing $\nu(\epsilon, n_0)$ by its mean value $\langle \nu(\epsilon) \rangle$, we obtain the same representation $I(E) - i\pi\nu(E)$ as in Eq. (33), with a predictable behavior of $I(E)$ and $\nu(E)$ (Fig. 5b).

Weak impurities. In this case, a behavior of functions $I(E)$ and $\nu(E)$ differs from their behavior in an ideal crystal by small smoothening of the Van Hove singularities (Fig. 5b). Dependence on n_0 results in fluctuations of the form of these functions, which are also small. It is clear that for weak impurities ($|V| \ll J$) the resonance condition is not fulfilled and no localization of the order parameter is possible.

For the delocalized regime, it is convenient to present the result (61) in another form. Taking the one-impurity configuration $\Delta(r) = \Delta_0 + \Delta_0 f(r)$, $\nu(r) = \nu_0 + \nu_1(r)$ and substituting it into equation (10), we have for the effective density of states entering into the BCS formula:

$$\nu_{\text{eff}} = \frac{\nu_0 + \nu_0 \langle f \rangle + \langle \nu_1 \rangle + \langle \nu_1 f \rangle}{1 + \langle f \rangle}. \tag{62}$$

Subtracting the result with $f \equiv 0$ and retaining the main terms in L^{-d}:

$$\nu_{\text{eff}} - \langle \nu \rangle = \langle \nu_1 f \rangle = \frac{\langle \nu_1^2 \rangle}{\nu_0} = \frac{\nu_0 (\mathcal{T}')^2}{4 k_F L^3}, \qquad \langle \nu \rangle - \nu_0 = \langle \nu_1 \rangle = -\frac{\nu_0 \mathcal{T}'}{2 k_F^2 L^3}, \tag{63}$$

where we have taken into account that only the term with $\mathcal{T}' \approx V + V^2 I(E_F)$ is essential in Eq. (46) for weak

[14] For weak disorder, relation $\Delta_1(r) = \Delta_0 \nu_1(r)/\nu_0$ follows from the second equation (41) after neglecting the quantity $F(r)$, which is of the second order. Its validity for the delocalized regime without assumption of small $\nu_1(r)$ is a non-trivial result expressed by equation (48).

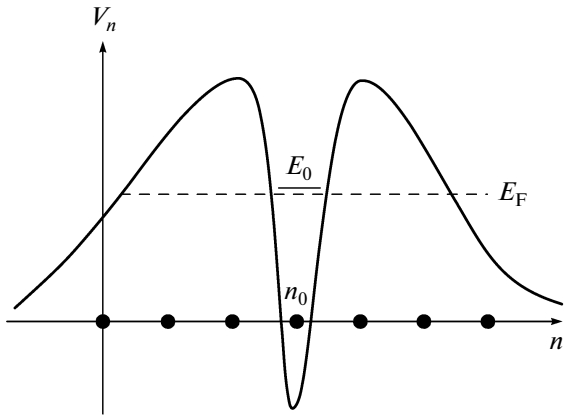

Fig. 8. A typical fluctuation of the random potential responsible for existence of the quasi-local state in the deep of the allowed band.

impurities. Inserting impurities one after another and averaging over V,

$$\nu_{\text{eff}} - \langle \nu \rangle \sim c\nu_0 \frac{W^2}{J^2}(k_F a)^{-1}, \qquad \langle \nu \rangle - \nu_0 \sim c\nu_0 \frac{W^2}{J^2}(k_F a)^{-2}, \tag{64}$$

we see that in the course of increasing a concentration, the increment of the quantity $\nu_{\text{eff}} - \langle \nu \rangle$ is by a factor $k_F a$ smaller than the increment of $\langle \nu \rangle - \nu_0$. Near the band edge one have $\langle \nu \rangle \gg \nu_0$ for sufficiently large concentrations, so $\nu_{\text{eff}} - \langle \nu \rangle \sim k_F a \langle \nu \rangle$ and deviations from the Anderson theorem are small. Near the band center we have $k_F a \sim 1$ and differences $\langle \nu \rangle - \nu_0$ and $\nu_{\text{eff}} - \langle \nu \rangle$ are small till concentrations $c \sim 1$; so $\nu_{\text{eff}} - \langle \nu \rangle \ll \langle \nu \rangle$. It is clear that violation of self-averaging for the order parameter does not occur for weak impurities.

In the 3D case, isolated weak impurities do not produce bound states beyond the initial spectrum (it is clear from Fig. 5a), and a finite density of states in this energy interval is a collective phenomenon related with long-range fluctuations of the band edge. Consider a fluctuation in the region of size L, due to which the range of V values is somewhat restricted, $(-W, W - 2\delta)$ instead $(-W, W)$. Then the mean value of the random potential is decreased by a quantity δ, while a probability of such fluctuation $\exp(-L^d\delta/W)$ is not small for $L^d\delta/W \lesssim 1$. Such fluctuations occur at all scales and produce a finite density of states beyond the bare spectrum.[15] If size L of the fluctuation is sufficient for existing of superconductivity, the latter will not differ from superconductivity in the initial system with not shifted band edge; i.e., impurities will not violate the uniformity of the order parameter. Near the initial band edge, the indicated fluctuations strongly overlap and superconductivity is quasi-homogeneous. Such fluctuations become spatially isolated in the region of strong localization, where they can be described in terms of the size effect (Section 6).

Strong impurities. For a small concentration of strong impurities ($|V| \gtrsim J$), a behavior of functions $I(E)$ and $\nu(E)$ is not very different from their behavior in an ideal crystal. However, in the region of the maximum of $I(E)$ the density of states becomes finite and, at first glance, complicates the occurrence of resonances. In fact, a new phenomenon comes to life. Since now $G^0_{n_0 n_0}$ depends on n_0, the attenuation of the quasi-local state will be determined not by average density of states, but its local value at the point n_0, which can be small in a fluctuational manner. As a result, resonances become possible even for energies in the deep of the band, where they were forbidden in the ideal lattice. The typical situation, when the local density of states $\nu_F(n_0)$ is small, corresponds to large values of the random potential in the vicinity of n_0; if now an impurity with large negative V is inserted into the site n_0, then a specific resonance configuration arises (Fig. 8).[16] In the "minimal" variant, such configuration corresponds to existence of large barriers at the nearest neighbours of site n_0, while a value of the potential at n_0 is chosen so that a corresponding level was in the interval of width $\sim T_c$ near the Fermi energy (the probability of this event is $\sim T_c/W$). For a finite band, both large positive and large negative value of the potential are locking, and for $W \gg J$ such values occur with probability p close to unity. Therefore, the probability of the "minimal" fluctuation

$$P_{\text{res}} \sim p^Z T_c/W, \tag{65}$$

where Z is a number of the nearest neighbours. It is clear that such resonances can occur for any position of the Fermi level. In the region of the fluctuational tail, the density of states is small by the natural reasons and there is no need to create the barrier around n_0; so the factor p^Z will be absent but the less probable form of the effective potential well is necessary, in order the

[15] The amplitude of long-range fluctuations can be seen from the fact that in the extremal cases the whole band is shifted by a quantity W or $-W$, i.e. such fluctuations by themselves (with no account for partial discretization of spectrum) cannot produce unbounded values of $\nu_F(r)$.

[16] According to [28], such configurations are responsible for multifractal statistics. It appears, that the tails of the distribution function are determined by individual peaks (and not fractal clusters), in correspondence with our conception. Thereby, we do not ignore the existence of multifractality but give another description of its influence on superconductivity.

level was in the desired part of the spectrum.[17] With increasing of the impurity concentration, the effective bandwidth is extended and the maximum of $I(E)$ is shifted correspondingly. However, the general mechanism of resonances and estimation of their probability remain unchanged.

Since the true critical temperature is hardly observable, it is actual to consider the "bulk T_c," which can be defined as T_c of the system with excluded resonant impurities. For strong but not resonant impurities, two terms in Eq. (48) are of the same order (see (50), (52)), and impurities are independent till concentrations $c \sim 1$, since the mobility edge lies far from the bare edge of spectrum and $k_F a \sim 1$. Validity of the Anderson theorem holds on the qualitative level: T_c is determined by the effective density of states, which differs from the average one by a factor of the order of unity.

6. SIZE EFFECT IN THE LOCALIZED PHASE

In the localized phase, the system breaks up into quasi-independent blocks of size ξ, and superconductivity is suppressed due to the size effect. Below we analyze this effect in terms of the Gor'kov equation. Superconductivity in small samples was discussed in many papers (see a review article [29]), but this discussion mainly concerns the aspects:

(a) inadequacy of the grand canonical ensemble due to a fixed number of electrons in small granules;

(b) parity effects;

(c) insufficiency of the mean field approximation;

(d) absence of an abrupt phase transition, etc. which are essential for finite systems and completely not actual in the present context. In principle, it is correct to stress unreliability of the mean field approach, but all attempts to overcome it (from modified mean field approximations till the exact Richardson solution and a direct numerical modelling) are based on the truncated BCS Hamiltonian, which by itself induces the certain way of pairing (in general incorrect).[18] As for the Gor'kov equation, it corresponds to the saddle-point approximation in the functional integral [30, 31] and is the most grounded of all mean-field type approaches; in addition, the electron interaction is specified in the physically clear manner and independently of one-electron states (Section 2). The accuracy of approximation is determined by the Ginzburg parameter, which provides insignificance of fluctuations in case of a superconductor (with exception of some special cases: e.g., in finite systems fluctuations have a qualitative importance, destroying a phase transition). The Gor'kov equation can be also obtained from the Eliashberg equations in the limit of the local interaction [14].

Consider the cubic sample of size L, accepting the periodical boundary conditions for the electron eigenfunctions. In a pure superconductor the latter have a form of plain waves, so $|\varphi_s(r)|^2 = L^{-d}$ and the local density of states (8) does not depend on r. Then $\Delta(r) = \text{const}$ is an exact solution of the Gor'kov equation (4), which reduces to

$$\Delta = gT\sum_{\omega} L^{-d}\sum_{s}\frac{1}{\epsilon_s^2+\omega^2}\Delta \tag{66}$$

and coincides with (9) in case of the continuous spectrum. In a small energy interval, the spectrum can be considered as a set of equidistant levels with a spacing Ω

$$\epsilon_s = \Omega(s+1/2), \quad \Omega = 1/\nu_F L^d, \tag{67}$$

where we accept that the Fermi energy lies in the middle between two discrete levels.[19] Substitution to (66) and summation over s gives

$$1 = gT\sum_{\omega}\frac{\pi\nu_F}{|\omega|}\tanh\frac{\pi|\omega|}{\Omega}. \tag{68}$$

For small Ω, the argument of the hyperbolic tangent is large and one can set $\tanh x = 1 - 2e^{-2x}$, so

$$\frac{1}{g\nu_F} = \ln\frac{1.14\omega_0}{T} - 4e^{-2\pi^2 T/\Omega}, \tag{69}$$

where we retained only main terms with $\omega = \pm\pi T$ in the second sum over ω. Subtracting the analogous equation with $\Omega = 0$, it is easy to obtain

$$T_c = T_{c0}[1 - 4e^{-2\pi^2 T_{c0}/\Omega}], \quad \Omega \ll T_{c0}. \tag{70}$$

For $T \longrightarrow 0$, one can replace summation in (68) by integration and obtain the equation for the critical value of Ω, at which superconductivity is destroyed

$$\frac{1}{g\nu_F} = \int_{-\omega_0}^{\omega_0}\frac{d\omega}{2\omega}\tanh\frac{\pi\omega}{\Omega_c} = \ln\frac{\pi\omega_0}{\Omega_c} - \int_0^{\infty}\frac{\ln x}{\cosh^2 x}dx. \tag{71}$$

The last integral is equal $\ln(\pi/4\gamma)$, where $\ln\gamma = C = 0.577$ is the Euler constant and comparing with the result for T_{c0}

$$1 = g\nu_F\ln\frac{4\gamma\omega_0}{\Omega_c}, \quad 1 = g\nu_F\ln\frac{2\gamma\omega_0}{\pi T_{c0}}, \tag{72}$$

[17] Strictly speaking, the resonant configurations of such kind are possible for small W in the vicinity of the initial band edge. However, a size of such configurations is inevitably large (due to restriction of the barrier height and absence of levels in a shallow well of a small radius), so they have a negligible probability and are incompatible with electroneutrality (Section 3).

[18] The state φ_s is coupled with its complex conjugated: it is correct only for a uniform order parameter [20].

[19] Such assumption is commonly accepted [29] for the case of the even number of electrons N; for add N it is accepted $\epsilon_s = \Omega s$, but the level $\epsilon_0 = 0$ is considered as "blocked," i.e., occupied by the unpaired electron and not participating in the scattering process. In the latter case, the results are analogous but correspond to smaller T_c.

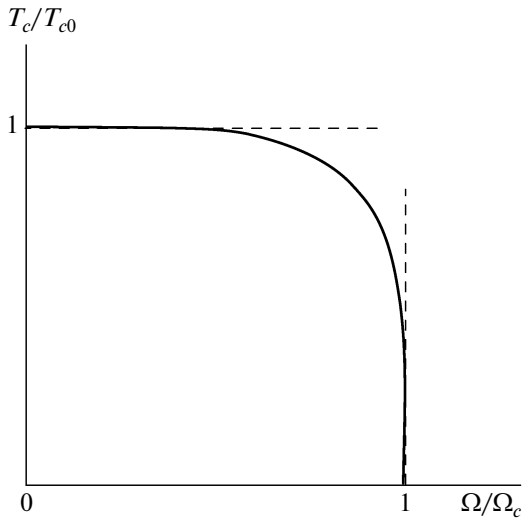

Fig. 9. "Rectangular" dependence of T_c on the level spacing Ω in a finite system; it is universal in the reduced coordinates $y = T_c/T_{c0}$, $x = \Omega/\Omega_c$.

one can see that

$$\Omega_c = 2\pi T_{c0}. \tag{73}$$

To find the dependence of T_c on Ω in the vicinity of Ω_c, one can transfer (68) using the Poisson summation formula [32]

$$\frac{1}{g\nu_F} = \sum_{s=-\infty}^{\infty} e^{-i\pi s} \int_{-\pi\omega_0/\Omega}^{\pi\omega_0/\Omega} dx \frac{\tanh x}{2x} \exp\left\{ i\frac{s\Omega}{\pi T} x \right\}, \tag{74}$$

where the term with $s = 0$ corresponds to (71). For $s \neq 0$, the integrals are convergent at large $|x|$ and it is possible to set $\omega_0 = \infty$ in them. Due to evenness in s they can be calculated for $s > 0$; then the contour is shifted in the upper half-plain and the main contribution arises from the pole $x = i\pi/2$. For $\Omega/T \gg 1$ it is sufficient to retain the terms with $s = 0, \pm 1$,

$$\frac{1}{g\nu_F} = \ln\frac{4\gamma\omega_0}{\Omega} - 4e^{-\Omega/2T}, \tag{75}$$

and subtracting the analogous equation with $T = 0$

$$T_c = \frac{\Omega_c}{2\ln[4\Omega_c/(\Omega_c - \Omega)]}, \quad \Omega \longrightarrow \Omega_c. \tag{76}$$

In the reduced coordinates

$$y = T_c/T_{c0}, \quad x = \Omega/\Omega_c \tag{77}$$

one can obtain the universal dependence $y(x)$. Indeed, transforming (68) by subtraction of the analogous equation with $\Omega = 0$, one has

$$\ln\frac{T}{T_{c0}} = T\sum_{\omega} \frac{\pi}{|\omega|}\left(\tanh\frac{\pi|\omega|}{\Omega} - 1 \right), \tag{78}$$

where ω_0 can be set to infinity. Substituting the Matsubara values $\pi T(2n + 1)$ for ω, one can present the dependence $y(x)$ in the parametric form

$$y = \exp F(t), \quad x = t^{-1}\exp F(t),$$
$$F(t) = 2\sum_{n=0}^{\infty} \frac{1}{2n+1}\left[\tanh\frac{\pi(2n+1)t}{2} - 1 \right], \tag{79}$$

where t runs from zero to infinity. Numerical calculation based on (79) gives the "rectangular" dependence $y(x)$ shown in Fig. 9: this dependence has exponentially small deviation from the horizontal line near $y = 1$, and exponentially small deviation from the vertical line near $x = 1$.

The given consideration retains for a disordered superconductor if possibility of self-averaging is accepted.[20] The obtained results can be used to describe the dependence of T_c on the distance to the mobility edge in the localized phase, where the system is divided into quasi-independent blocks of size ξ. The role of Ω is played by the quantity

$$\Omega(E) \sim J(\xi/a)^{-d} \sim J(|E - E_c|/J)^{d\nu}, \tag{80}$$

where ν is the critical exponent of the localization length. According to Section 5, the assumption of self-averaging is valid literally for weak disorder and on the qualitative level for strong disorder in the absence of resonances. In the latter case, T_c is determined by the effective density of states which differs from the average one by a factor of the order of unity, which is a smooth function of parameters. It preserves the character of singularities (70) and (76), which determine the behavior near E_c and E^* (Fig. 3) and are responsible for the most striking features in the dependence $T_c(\epsilon_F)$.

7. CONCLUSIONS

The present paper resolves contradiction between two series of papers [1–5] and [9, 10]. The obtained results has in some way a compromise character. On the one hand, the "bulk" superconductivity behaves in correspondence with the picture by Bulaevskii and Sadovskii [1–5]. On the other hand, the true transition temperature T_c of strongly disordered superconductor does not coincide with the "bulk" one and is determined by rare peaks of the order parameter on the atomic scale; in correspondence with [9, 10] it has a power law dependence on the coupling constant and does not depend on the cut-off frequency. However, in contrast to [9, 10], it has no essential dependence on the position of the Fermi level and does not correlate with the Anderson transition. By this reason, we do

[20] Of course, in this case one should take some realistic statistics of the Wigner–Dyson kind instead of the equidistant levels, but it has a small effect on the results [29].

not see any grounds to say on "fractal superconductivity" [10] near the localization threshold.

The presented results are obtained in the framework of the mean field theory, which is surely valid in the delocalized regime. In fluctuational theory, essential modification of results is expected only for the localized regime: the modulus of the order parameter changes slightly, while fluctuations of its phase become essential. We should stress that the role of fluctuations is determined by specific values of parameters, characterizing the system: if, for example, the ratio T_c/J is not too small, then the resonant impurities have rather large concentration and the Josephson coupling between the localized superconducting "drops" is sufficiently large for stabilization of the mean-field solution (this coupling is determined mainly by existence of the uniform contribution (see (30)), which grows at small L). Contrary, if $T_c/J \longrightarrow 0$, then the Josephson coupling between drops is small and fluctuations essentially suppress T_c. According to the nonlinear Ginzburg–Landau equations derived in [11] for the localized regime, decreasing of the temperature stimulates the growing of tails of the localized solutions; the Josephson coupling between drops becomes greater and stabilizes the mean-field solution before the "bulk T_c" is reached. Analogous remarks are valid in relation with the Coulomb blockade effects [30].

In comparison of the obtained results with experiment, one should have in mind, that the continuous distribution $P(V)$ in the Anderson model is not very realistic; it is more adequate to assume the discrete (and not very dense) set of the V values. As a result, in most systems the described resonances will be unobservable for any concentration and arrangement of impurities. However, in the minority of systems the effect of resonances will be strong and stable. The Anderson model with a several types of periodically arranged impurities can be considered as the model for the high-temperature oxide superconductors. The possibility to interpret the "superconducting explosion" of 1987 as localization of the order parameter was indicated previously [11]; the above results suggests possibility of such localization not only in the Cu–O planes but also at the individual atoms. The adequacy of such a model is confirmed by (a) optimistic estimates of T_c, (b) practical coincidence of the maximal T_c values with ω_0/π, (c) suppressed isotope-effect in the regime $\epsilon_c \ll \omega_0$.

APPENDIX

On Solution of the Gor'kov Equation with the Kernel (37)

Let fill in the gaps for our exposition in the main text.

In the region remote from the resonance, we can consider $\mathcal{T}'_\omega$ and $|\mathcal{T}''_\omega|$ as independent of ω: then Z_ω is also constant. Substituting $\Delta_1(r)$ from (43b) into expressions (44) for X_ω and Y_ω, we have representation (47) with parameters

$$X = \Delta_0 \ln\frac{1.14\omega_0}{T}\lambda_{01} + \frac{\omega_0}{\pi}\lambda_{11}Z,$$
$$Y' = \Delta_0 \ln\frac{1.14\omega_0}{T}\lambda'_{02} + \frac{\omega_0}{\pi}\lambda'_{12}Z, \qquad (A.1)$$
$$Y'' = \Delta_0 \ln\frac{1.14\omega_0}{T}\lambda''_{02} + \frac{\omega_0}{\pi}\lambda''_{12}Z.$$

Then Z_ω has a form

$$Z = \Delta_0 \ln\frac{1.14\omega_0}{T}B_2 + Z\frac{\omega_0}{\pi}B_3, \qquad (A.2)$$

and its combination with (43a) gives a system of equations for Δ_0 and Z

$$\Delta_0\left[-\frac{1}{B_1} + g\langle\nu_1\rangle_0 + g^2\langle\nu_1^2\rangle_0 \ln\frac{1.14\omega_0}{T}\right]$$
$$+ g\frac{\omega_0}{\pi}\lambda_{01}Z = 0, \qquad (A.3)$$
$$\Delta_0 B_2 \ln\frac{1.14\omega_0}{T} + Z\left[-1 + \frac{\omega_0}{\pi}B_3\right] = 0,$$

with the coefficients

$$B_1 = \frac{K_0(0)}{\lambda_0 L^3 \tau}\ln\frac{1.14\omega_0}{T},$$
$$B_2 = 2\mathcal{T}'\lambda'_{02} - 2\mathcal{T}''\lambda''_{02} + |\mathcal{T}|^2\lambda_{01}, \qquad (A.4)$$
$$B_3 = 2\mathcal{T}'\lambda'_{12} - 2|\mathcal{T}''|\lambda''_{12} + |\mathcal{T}|^2\lambda_{11}.$$

The terms containing ω_0 has an additional smallness $\sim\omega_0/J$ and can be neglected;[21] the condition of solubility for (A.3) gives the result (48).

In the vicinity of the resonance, one cannot neglect the ω dependence of the quantities $\mathcal{T}'_\omega$, $|\mathcal{T}''_\omega|$, and consequently Z_ω. Substituting $\Delta_1(r)$ from (43b) into (44) for X_ω and Y_ω, one has representation (47) with parameters

$$X = \Delta_0 \ln\frac{1.14\omega_0}{T}\lambda_{01} + \lambda_{11}S,$$
$$Y' = \Delta_0 \ln\frac{1.14\omega_0}{T}\lambda'_{02} + \lambda'_{12}S, \qquad (A.5)$$
$$Y'' = \Delta_0 \ln\frac{1.14\omega_0}{T}\lambda''_{02} + \lambda''_{12}S,$$

[21] We have in mind the traditional superconductors. If $\omega_0 \sim J$, then the "vicinity of the resonance" is extended and in fact occupies the whole band.

and for Z_ω

$$Z_\omega = \Delta_0 \ln\frac{1.14\omega_0}{T}[2\mathcal{T}'_\omega\lambda'_{02} - 2|\mathcal{T}''_\omega|\lambda''_{02} + |\mathcal{T}_\omega|^2\lambda_{01}] + S[2\mathcal{T}'_\omega\lambda'_{12} - 2|\mathcal{T}''_\omega|\lambda''_{12} + |\mathcal{T}_\omega|^2\lambda_{11}]. \quad (A.6)$$

Substitution into expressions (58) and (43) gives a system of equations for Δ_0 and S

$$\Delta_0\left[-\frac{1}{B_1} + g\langle\nu_1\rangle_0 + g^2\langle\nu_1^2\rangle_0\ln\frac{1.14\omega_0}{T}\right] + g\lambda_{01}S = 0,$$
$$\Delta_0 C_1\ln\frac{1.14\omega_0}{T} + S[-1 + C_2] = 0, \quad (A.7)$$

with definitions

$$C_1 = 2\lambda'_{02}\sigma_1 - 2\lambda''_{02}\sigma_2 + \lambda_{01}\sigma_3,$$
$$C_2 = 2\lambda'_{12}\sigma_1 - 2\lambda''_{12}\sigma_2 + \lambda_{11}\sigma_3,$$
$$\sigma_1 = T\sum_\omega\mathcal{T}'_\omega, \quad \sigma_2 = T\sum_\omega|\mathcal{T}''_\omega|, \quad (A.8)$$
$$\sigma_3 = T\sum_\omega|\mathcal{T}_\omega|^2.$$

The condition of solubility for the system (A.7) gives

$$\left(\frac{1}{B_1} - g\langle\nu_1\rangle_0 - g^2\langle\nu_1^2\rangle_0\ln\frac{1.14\omega_0}{T}\right)(-1 + C_2) + C_1 g\lambda_{01}\ln\frac{1.14\omega_0}{T} = 0. \quad (A.9)$$

Estimations for $\epsilon_0 \sim \gamma \sim T$ give

$$\sigma_1 \sim VE_0, \quad \sigma_2 \sim VE_0\ln\frac{\omega_0}{T}, \quad \sigma_3 \sim \frac{(VE_0)^2}{T}, \quad (A.10)$$

and allow to retain only the leading terms in J/T; as a result, Eq. (A.9) can be written as

$$\left(\tau - \frac{g^2\nu_0\langle\nu_1\rangle_0 + g^2\langle\nu_1^2\rangle_0}{\lambda_0^3 L^3}\right)(-1 + \lambda_{11}\sigma_3) + \frac{\lambda_{01}^2}{\lambda_0^3 L^3}g\sigma_3 = 0, \quad (A.11)$$

and reduces to a form (59); the last term is essential only near the intersection point of dashed lines in Fig. 7, when $\epsilon_0 \sim \gamma \sim \epsilon_c$ and T should be replaced by ϵ_c in the estimates (A.10).

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